Vertical chaos and horizontal diffusion in the bouncing-ball billiard

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The bouncing-ball billiard is a low-dimensional system with which transport properties of real physical systems can be studied theoretically. We study the bouncing-ball billiard with nonconvex scatterers and small slopes. We show that between the horizontal and vertical motion there is a separation of time scales, which is controlled by the slope of the billiard. We apply the theory of time-scale separation developed by Kantz *et al.* Physica D **187**, 200 (2004). If the vertical motion is chaotic, the horizontal motion is diffusive, but if the vertical motion is (quasi)periodic, there is no diffusion. We confirm the results with numerical simulations. Hence, the order-chaos transition in the vertical degrees of freedom translates into a localization-delocalization transition for the horizontal motion.

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I. INTRODUCTION

The concept of a classical heat bath, i.e., a large reservoir of degrees of freedom, being coupled to a system of interest has a long tradition. One can show, using methods such as those outlined by van Kampen [1] or Fick and Sauermann [2], that under suitable conditions a deterministic (even Hamiltonian) system coupled to a large number of again deterministic (even Hamiltonian) degrees of freedom can be modeled as being an open system subject to stochastic forces plus damping. In Refs. [3–5] it was shown that a similar reduction is also valid if the system of interest is coupled to only a few degrees of freedom, which however, are chaotic and much faster than the system of interest. Hence, in the presence of a pronounced time-scale separation, the fast chaos actually acts like a thermodynamic heat bath with stochastic driving and damping, but with clear signatures of the finite energy content.

We employ this concept here in order to study transport in a periodic potential. We will discuss a single particle that is bouncing on a periodically vibrating surface with a sawtooth-shaped contour. Depending on whether this vertical motion is chaotic or periodic, the particle will or will not diffuse in the horizontal direction. This study is of interest both from the view of its potential relevance for more realistic systems and for fundamental reasons.

In the study of granular media, systems called *vibratory conveyors* are studied and even used in practice: granular particles are placed on a vibrating surface, where for suitable surface structure and vibratory excitations the granular material moves in a controlled way [6]. Not all details responsible for the transport process have yet been understood; for an overview of recent results see [7]. On length scales that are many orders of magnitude smaller, similar effects are known. In the oxidation of CO by O_2 on Pt surfaces, it has been observed experimentally that CO molecules perform diffusive motion on the Pt surface whereas O_2 molecules stick to the place where they adhere to the surface [8]. The surface is subject to thermal excitations, which should act on both types of molecule in the same way, and also both molecules are composed of two atoms. Still, the internal molecular vibrations and also the binding energies between molecules and surface differ, hence giving rise to the different behavior. Even if this is a quantum-mechanical problem, the bouncing-ball model suggests that the difference in behavior might be related to whether or not the internal degrees of freedom of the molecules are chaotic.

Though the model we are studying here does not contribute in any way to a quantitative understanding of either of the two above-mentioned transport phenomena, it could be of primary relevance for their understanding. We will demonstrate that the chaoticity of the vertical motion relative to the surface is essential for diffusion in horizontal direction. In fact, we will first show that there is a time-scale separation between vertical and horizontal dynamics. This has the consequence that the vertical degrees of freedom could be eliminated and could be replaced by a suitable white noise process, if they are chaotic. If they are periodic, instead one could use the averaging principle [9] for their elimination, which, however, leads to deterministic equations for the slow degrees of freedom, so that the horizontal motion can be either periodic or ballistic, but not diffusive. Numerical simulations of the full system then show that indeed only if the fast degrees of freedom are chaotic does diffusion take place. We relate diffusion coefficients and Lyapunov exponents to each other and verify that the coupling between the slow and the fast subsystems introduces a damping in the effective stochastic motion of the slow system alone.

The principal aspect of this study is that the theory to which we will refer has until now only been elaborated for time-continuous autonomous systems. The bouncing-ball dynamics suggests that an extension of this theory to discrete time dynamics should be possible, and that relations obtained in the time-continuous case might be translated into the time-discrete case in a straightforward way. Moreover, the bouncing-ball system enables us to search for signatures of nonexponential decay of correlations in the fast subsystem, which would translate itself into anomalous diffusion in the horizontal direction. We find that, without perfect fine tuning of parameters, no anomalous diffusion exists, and hence conclude that normal diffusion (and therefore the description of

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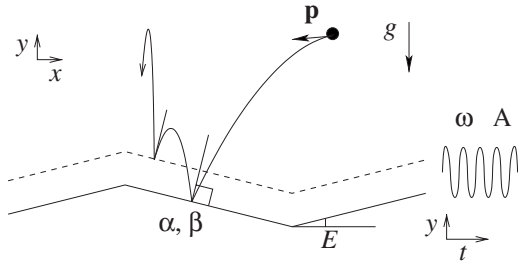

FIG. 1. Diagram of the bouncing-ball billiard.

the system by a Fokker-Planck equation) is very robust against nonhyperbolic structures in the fast phase space.

This paper is organized as follows. First, we will introduce the bouncing-ball billiard and explain its dynamics in Sec. II. Then, in Sec. III, we will describe time-scale separation, its consequences, and how it is present in the bouncing-ball billiard. Quantitative analytical expressions for diffusion and drift are derived. In order to confirm the predicted relation between transport and dynamical properties, numerical simulations of the bouncing-ball billiard were performed, which are described in Sec. IV. Section V contains a comparison between numerical estimates of the transport in the system derived from the theory on the one hand and the simulations on the other. Finally, in Sec. VI, general conclusions are drawn from the results.

II. MODEL

Consider a particle that is bouncing, elastically or inelastically, on a vibrating surface, while subject to a gravitational field. If the surface is at least one dimensional and has a translational symmetry perpendicular to the gravitational field, the system is referred to as a bouncing-ball billiard. A diagram of the bouncing-ball billiard is shown in Fig. 1. In this paper, we consider the two-dimensional bouncing-ball billiard with a sawtooth-shaped surface with absolute slope E, which is sinusoidally driven with angular frequency ω and amplitude A.

The bouncing-ball billiard is an extension of the one-dimensional problem of a ball bouncing inelastically on a vibrating surface. The bouncing-ball problem has been studied thoroughly experimentally [10,11] as well as theoretically [12–15]. Even in one dimension, the dynamics are so complicated that, in order to obtain exact analytical results, approximations are often needed, such as the high-bounce approximation, which leads to the dissipative standard map [11,16].

The dynamics of the bouncing ball are simple to write down. The phase space consists of the horizontal and vertical position of the particle, denoted by $\mathbf{r}=(x,y)$, and the corresponding momenta $\mathbf{p}=(p_x,p_y)$. The evolution in phase space consists of a sequence of alternating free flights and collisions with the surface. During a free flight the particle is only subject to the gravitational field $\mathbf{g}=(0,g)$ in the negative vertical direction ($g<0$). Without loss of generality, we may set the mass of the particle to unity. During the free flight following collision n at time $t^{(n)}$,

$$\mathbf{p}(t) = \mathbf{p}(t^{(n)}) + (t - t^{(n)})\mathbf{g}, \tag{1}$$

$$\mathbf{r}(t) = \mathbf{r}(t^{(n)}) + (t - t^{(n)})\mathbf{p}(t^{(n)}) + \frac{1}{2}(t - t^{(n)})^2\mathbf{g}. \tag{2}$$

The time $t^{(n+1)}$ at which the next collision occurs can be determined by intersecting the free-flight trajectory with the vibrating surface,

$$y(t^{(n)}) + (t^{(n+1)} - t^{(n)})p_y(t^{(n)}) + \frac{1}{2}(t^{(n+1)} - t^{(n)})^2 g = h(x(t^{(n+1)}), t^{(n+1)}), \tag{3}$$

where $h(x,t)$ denotes the height of the surface at horizontal position x at time t.

At a collision, momentum is exchanged between the moving surface and the particle, and the momentum of the particle is reflected, with certain restitution coefficients. Let primes denote the momenta after the collision. Let α be the restitution coefficient parallel to the surface and β the restitution coefficient perpendicular to the surface. If $\hat{\mathbf{n}}$ is the two-dimensional unit vector orthogonal to the plane of the surface, and $\mathbf{v}(t)$ is the velocity of the surface at time t,

$$\mathbf{p}' = \mathbf{v}(t) + [\alpha \mathbf{1}_2 - (\alpha + \beta)\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}] \cdot [\mathbf{p} - \mathbf{v}(t)], \tag{4}$$

where $\otimes$ indicates a tensor product, and $\cdot$ is the standard matrix product. The two-dimensional unity matrix is denoted by $\mathbf{1}_2$.

III. TIME-SCALE SEPARATION THEORY

Consider a system with continuous time that consists of two subsystems with coordinate vectors $\mathbf{a}$ and $\mathbf{b}$ which have two different time scales. The equations of motion can be written as

$$\dot{\mathbf{a}} = \mathbf{F}(\mathbf{a},\mathbf{b}), \tag{5}$$

$$\dot{\mathbf{b}} = \frac{1}{\epsilon}\mathbf{G}(\mathbf{a},\mathbf{b}), \tag{6}$$

where ϵ is a small parameter which quantifies the separation of the two time scales.

In Refs. [3–5] it was derived that the dynamics of the slow system can, under conditions of mixing and exponential decay of correlation, be well approximated by a linear Fokker-Planck equation with normal diffusion and drift. That is,

$$\frac{\partial \rho}{\partial t}(\mathbf{a},t) = +\sum_{\mu\nu} \frac{\partial}{\partial \mathbf{a}_\mu}\frac{\partial}{\partial \mathbf{a}_\nu} D^{(2)}_{\mu\nu}\rho(\mathbf{a},t) - \sum_{\mu} \frac{\partial}{\partial \mathbf{a}_\mu}[D^{(1)}_{\mu}(\mathbf{a})\rho(\mathbf{a},t)], \tag{7}$$

where $\rho(\mathbf{a},t)$ is the effective probability density of the slow coordinates, $D^{(1)}(\mathbf{a})$ is the effective drift vector, and $D^{(2)}$ is the effective diffusion matrix. The subscript μ indicates the various components of the slow coordinates.

The drift can be determined from the average coupling and a viscous damping term related to the decay of correla-

tion. The diffusion constant can also be determined from the coupling and the decay of correlation. Let $\eta(t',\mathbf{b}_0;\mathbf{a})$ denote a solution of Eq. (6) for fixed $\mathbf{a}$ and with initial condition $\mathbf{b}_0$, so that

$$\frac{\partial \eta[t,\mathbf{b};\mathbf{a}]}{\partial t} = \mathbf{G}(\mathbf{a}, \eta[t,\mathbf{b};\mathbf{a}]). \tag{8}$$

Let also $\langle\ \rangle$ indicate averaging with $\mathbf{b}$ taken as distributed according to the invariant density of Eq. (6) for fixed $\mathbf{a}$. Let $\delta\mathbf{F}$ be the function $\mathbf{F}(\mathbf{a},\mathbf{b})-\langle\mathbf{F}(\mathbf{a},\mathbf{b})\rangle$. The diffusion and drift can be written as

$$D^{(2)}_{\mu\nu}(\mathbf{a}) = \int_0^\infty dt' \langle \delta\mathbf{F}_\mu(\mathbf{a}, \eta(t'/\epsilon,\mathbf{b};\mathbf{a}))\, \delta\mathbf{F}_\nu(\mathbf{a},\mathbf{b})\rangle, \tag{9}$$

$$D^{(1)}_\mu(\mathbf{a}) = \langle \mathbf{F}_\mu(\mathbf{a},\mathbf{b})\rangle + \sum_\nu \int_0^\infty dt' \left\langle \mathbf{F}_\nu(\mathbf{a},\mathbf{b}) \frac{\partial \delta\mathbf{F}_\mu(\mathbf{a}, \eta(t'/\epsilon,\mathbf{b};\mathbf{a}))}{\partial \mathbf{a}_\nu} \right\rangle. \tag{10}$$

If the fast coordinates are in a periodic regime, then the diffusion vanishes, and the drift can be approximated by the average coupling.

A. Time-scale separation between horizontal and vertical coordinates

Let the height of the surface, denoted by $h(x,t)$, be a symmetric sawtooth in x with absolute slope E, and let E be small. The coupling between the horizontal and vertical positions and momenta is weak. If $1-\alpha$ is of order E, then the horizontal coordinates change one order of E more slowly from collision to collision than the vertical coordinates. There is therefore a time-scale separation between the horizontal and vertical coordinates.

The collisions in the bouncing-ball billiard are instantaneous. At the time of collision the force on the particle is infinite. Additionally, the bouncing-ball billiard is periodically driven, and therefore nonautonomous. As a consequence the continuous-time dynamics cannot trivially be written in the form of Eqs. (5) and (6). They can, however, be discretized and written in a similar form if the system is described directly after each collision. The dynamics can then be rewritten as a discretized version of Eqs. (5) and (6), with the horizontal coordinates as the slow variable and the vertical coordinates as the fast variable. The slope of the profile can be considered as a time-scale separation parameter.

The dynamics are described by a map which maps the position and momentum of the bouncing ball just after a collision with the surface to the position and momentum after the next collision with the surface. This implies a reduction of the phase space dimension from 5 (the particle position and momentum, and the phase of the vibration) to 4. The results are equations of the form

$$\mathbf{a}^{(n)} - \mathbf{a}^{(n-1)} = \tilde{\mathbf{F}}(\mathbf{a}^{(n-1)}, \mathbf{b}^{(n-1)}|t^{(n)}), \tag{11}$$

$$\mathbf{b}^{(n)} - \mathbf{b}^{(n-1)} = \frac{1}{\epsilon}\tilde{\mathbf{G}}(\mathbf{a}^{(n-1)}, \mathbf{b}^{(n-1)}|t^{(n)}), \tag{12}$$

where ϵ is the time-scale separation parameter. As long as $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are of the same order of magnitude, the time-scale separation holds. Time is treated as an extra parameter here. The functions $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ may contain an extra time-dependent rescaling factor without affecting the time-scale separation.

Equations (9) and (10) hold for continuous-time systems, yet in the bouncing-ball billiard the force exerted on the particle is not a continuous function. Therefore it is not directly clear whether Eqs. (9) and (10) are the correct expressions for the diffusion and drift. The rate of change per discrete time step is very small, and therefore the form of the Fokker-Planck equation remains unchanged. A drift exists with a viscous damping term, and diffusion is normal.

Some systems with discrete time can be viewed as continuous-time systems that are sampled at specific intervals. However, this does not hold for all systems, and, even if an extension to a continuous system exists, finding it is not straightforward. For some more details on time-scale separation in maps, see, for instance, Refs. [17,18].

1. Equations of motion

The geometry of the collision is determined by the vector $\hat{\mathbf{n}}$ normal to the surface. This vector can be written as a function of the slope E of the surface, to read

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{(1+E^2)}}(\pm E, 1), \tag{13}$$

where the $\pm$ is a $+$ if the slope is negative, and $-$ if the slope is positive. Let $\Delta t^{(n)}$ denote $t^{(n)}-t^{(n-1)}$. The map of the coordinates can then be written as

$$p_x^{(n)} - p_x^{(n-1)} = \left((\alpha-1)p_x^{(n-1)} - \frac{\alpha+\beta}{1+E^2}\{E^2 p_x^{(n-1)} \pm E[p_y^{(n-1)} - v_y(t^{(n)}) + g\Delta t^{(n)}]\}\right), \tag{14}$$

$$p_y^{(n)} - p_y^{(n-1)} = \left(v_y(t^{(n)}) + (\alpha-1)p_y^{(n-1)} + \alpha[g\Delta t^{(n)} - v_y(t^{(n)})] - \frac{\alpha+\beta}{1+E^2}[\pm E p_x^{(n-1)} + p_y^{(n-1)} - v_y(t^{(n)}) + g\Delta t^{(n)}]\right), \tag{15}$$

$$x^{(n)} - x^{(n-1)} = p_x^{(n)}\Delta t^{(n)}, \tag{16}$$

$$y^{(n)} - y^{(n-1)} = p_y^{(n)}\Delta t^{(n)} + \frac{1}{2}(\Delta t^{(n)})^2 g = O(E p_x^{(n)}\Delta t^{(n)} + A\ \max(1, \omega\Delta t^{(n)})). \tag{17}$$

The time $t^{(n+1)}$ at which the next collision occurs can be determined from Eq. (3). As long as the slope does not change rapidly from hop to hop, the rapid change of x in Eq. (16) does not affect the separation of the time scales. There-

fore, if the particle only hops across a kink in the sawtooth less than $1/E$ of its hops, Eq. (16) can be brought into the form of Eq. (11), by rescaling with a factor of $\max(1,\Delta t^{(n)})/E$. After rewriting the dynamics, the equations are of the form of Eqs. (11) and (12), if $\alpha-1$ is sufficiently small. The absolute value of the slope of the surface, E, gives the time-scale separation parameter ϵ.

2. Predictions for transport

The time-scale separation between the vertical and horizontal coordinates has a structure different from the usual types of time-scale separation between two coupled systems. It relates the dynamical properties of the vertical coordinates of the particle to the transport properties in the horizontal coordinates. The theory developed in Refs. [3–5] and summarized at the beginning of this section makes clear predictions for the behavior of the system. If the vertical coordinates are in a periodic orbit, then the horizontal coordinates behave deterministically with a drift that can be determined from the average coupling. If the vertical coordinates behave chaotically, and correlation in the vertical coordinates decays exponentially, then the horizontal coordinates will exhibit normal diffusion with the diffusion constant in Eq. (9) and a viscous damping term in the drift, shown in Eq. (10). Even if the restitution parallel to the surface is equal to unity, the diffusion is predicted to be normal. A position- and momentum-dependent diffusion on the surface of the profile would lead directly to normal diffusion in the well numbers. Note that, though the well number is directly related to the position, these are two different diffusion quantities. One is the formal diffusion tensor in the Fokker-Planck equation; the other is a scalar diffusion constant.

The relative rate of change in the horizontal momentum is of order E per collision. For $\alpha=1$, this change is even independent of the horizontal momentum. Conversely, the rate of change in the horizontal position is equal to the horizontal momentum, and therefore independent of the fast coordinates. It follows from Eqs. (8)–(10) that, if the fast motion is chaotic and $\alpha=1$,

$$D_{xx}^{(2)}(x,p_x)=0, \tag{18}$$

$$D_{xp}^{(2)}(x,p_x)=D_{px}^{(2)}(x,p_x)=0, \tag{19}$$

$$D_{pp}^{(2)}(x,p_x)\propto E^2+O(E^3), \tag{20}$$

$$D_{x}^{(1)}(x,p_x)=p, \tag{21}$$

$$D_{p}^{(1)}(x,p_x)=\pm O(E)[1+O(p_x^2)]+D_{\text{friction}}, \tag{22}$$

$$D_{\text{friction}}\propto -pE^2[1+O(E)O(p_x^2)]. \tag{23}$$

Together with the Fokker-Planck equation (7), these results describe the reduced motion of the slow system. The first term on the right-hand side of Eq. (22) is the average coupling. The second term originates from the second term on the right-hand side of Eq. (10) and is a viscous friction term. If the fast motion is (quasi)periodic, the diffusion tensor is 0, and the friction term is of higher order in E.

B. Peculiarities of mixed phase space and anomalous transport

The time-scale separation theory requires exponential mixing for the system to be described by the linear Fokker-Planck equation (7). This requirement may not always be met if the phase space is complicated by coexistence of stable periodic orbits and a chaotic sea.

1. Stable islands in a chaotic sea

The one-dimensional bouncing-ball problem has a complicated phase space, which, for some values of the parameters, contains several attractors with separate basins of attraction, including (quasi)periodic orbits surrounded by attracting islands. In such a mixed phase space, having both stable periodic orbits and chaotic trajectories, the trajectories η in Eqs. (8)–(10) depend on the initial conditions. As long as the resulting trajectories are either periodic or in a chaotic sea with exponential mixing, the system is still described by an effective diffusion and drift. The estimates for the diffusion and drift in Eqs. (9) and (10) are still valid, with averages over the appropriate subset of the phase space.

The dependence of the behavior of the vertical coordinates on initial conditions might lead to temporal correlations in the effective diffusion coefficient, because the system could become trapped near a periodic orbit in the vertical coordinates. If this correlation does not decay exponentially, the system is no longer described by a linear Fokker-Planck equation. In the bouncing-ball billiard with a sawtooth profile, such correlation is quickly destroyed when the particle hops from a positive to a negative slope.

2. Decay of correlation

Anomalous diffusion is not excluded by the time-scale separation. However, the system can be described by a linear Fokker-Planck equation only if correlation decays exponentially. For very specific parameter values, this may not be the case and the diffusion across the wells may be anomalous. This can occur if there is an unstable periodic orbit of the full system that is not linearly, but more weakly, unstable. Such orbits exist only at points where a periodic orbit becomes unstable, and do not exist for almost all parameter sets. Near such a marginally unstable periodic orbit correlations decay with a power law. In such a case, the dynamics can show intermittent behavior, staying near the periodic orbit for long, power-law-distributed times, and displaying irregular trajectories in between. Overall, the diffusion would become anomalous. If the trajectory of the periodic orbits is restricted to a single well, they could lead to subdiffusion. The periodic orbits could also lead to accelerator modes, hopping from well to well with a constant average horizontal velocity, which would lead to a ballistic motion in the horizontal coordinate. Such orbits could produce superdiffusion [19,20].

IV. SIMULATIONS

The time-scale separation implies a direct relation between the presence of chaos in the vertical coordinates and transport in the horizontal direction. Equations (18)–(23)

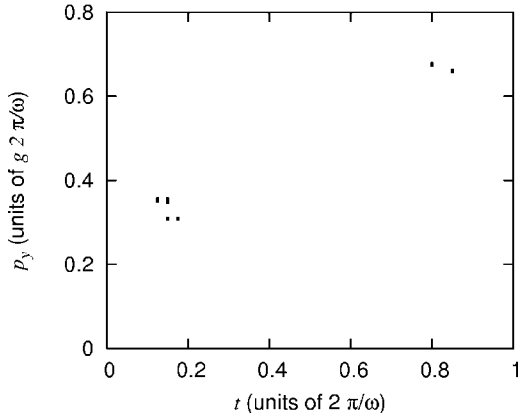

FIG. 2. Poincaré section with 500 000 points of a periodic trajectory with, in this case, period 24. The vertical velocity after the collision is plotted versus the collision time modulo $2\pi/\omega$. The system parameters are $4\pi^2 g/(\omega^2 A)=48.828$, $E=0.01$, $\alpha=0.999$, and $\beta=0.96$.

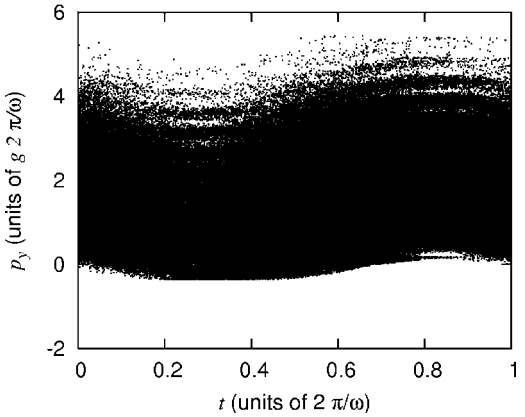

FIG. 3. Poincaré section with 500 000 points of a chaotic trajectory. The vertical velocity after the collision is plotted versus the collision time modulo $2\pi/\omega$. The parameters and initial conditions are the same as in Fig. 2, except for $4\pi^2 g/(\omega^2 A)=35.086$. The largest Lyapunov exponent is positive, $\lambda_1 \approx 0.428\omega/(2\pi)$.

give the quantitative behavior of the transport.

In order to confirm the validity of the time-scale separation theory, numerical simulations were performed for the bouncing-ball billiard and compared with the theoretical predictions. The dynamics were calculated from collision to collision, by numerically solving the transcendent equation for the collision times using the GNU Scientific Library. Lyapunov exponents were calculated using a reorthonormalization procedure. The exponent of the diffusion was extracted from the well number as a function of time.

Figures 2 and 3 show examples of Poincaré sections of two trajectories from numerical simulations with similar parameters, except for $4\pi^2 g/(\omega^2 A)$. In Fig. 2, the vertical coordinates are in a stable periodic orbit, and there is no horizontal diffusion. In Fig. 3, the vertical coordinates are in the chaotic regime. There is normal horizontal diffusion, with an exponent of 1.001 ± 0.002.

In the numerical simulations all chaotic systems exhibit normal diffusion. When the largest Lyapunov exponent is positive, the diffusion does not deviate significantly from normal diffusion. When the largest Lyapunov exponent is negative, there is no diffusion or drift. The largest Lyapunov exponent, the diffusion exponent, and the diffusion constant are plotted as a function of $4\pi^2 g/(\omega^2 A)$ in Fig. 4, for an example set of parameters. The initial conditions were kept the same for all systems. In some cases they are inside a stable periodic island, and for other frequencies they are in the chaotic sea. Figures 2 and 3 are both represented in Fig. 4. The largest Arnold tongues in the one-dimensional bouncing-ball problem have been studied in, for instance, Ref. [14].

The diffusion and drift can be calculated numerically from the trajectories of the full system by considering time averages $\langle\ \rangle_t$ of appropriate functions of changes in the coordinates $\Delta(x,p_x)$ over finite time intervals Δt. We have

$$\left\langle \left. \frac{\Delta(x',p_x')}{\Delta t} \right|_{x'=x,p_x'=p_x} \right\rangle_t = D^{(1)}(x,p_x) + O(\Delta t). \quad (24)$$

The higher-order terms in Δt are due to the fact that the variables are not constant during the time interval. Provided that correlation decays within the time interval,

$$\left\langle \left. \frac{[\Delta(x',p_x') - D^{(1)}(x',p_x')\Delta t] \otimes [\Delta(x',p_x') - D^{(1)}(x',p_x')\Delta t]}{\Delta t} \right|_{x'=x,p_x'=p_x} \right\rangle_t = D^{(2)}(x,p_x) + O(\Delta t). \quad (25)$$

The time intervals needed for correlation to decay will need to include several hops of the particle.

In our system, the drift in the x coordinate is quite large, of order p_x, whereas the drift in p_x is only of order ϵ. Consequently, the higher-order terms in Δt will be quite large for the diffusion in x, but one order in E smaller for the diffusion in p_x. Without including further corrections, which would require knowledge of the functional dependence of the diffusion and drift on p_x, it is therefore only possible to accurately extract the diffusion in the horizontal momentum, D_{pp}, from the trajectories.

V. ESTIMATES OF DIFFUSION AND DRIFT FROM THE TIME-SCALE SEPARATION THEORY

The results from the numerical simulations can be compared directly to the theoretical predictions from Eqs.

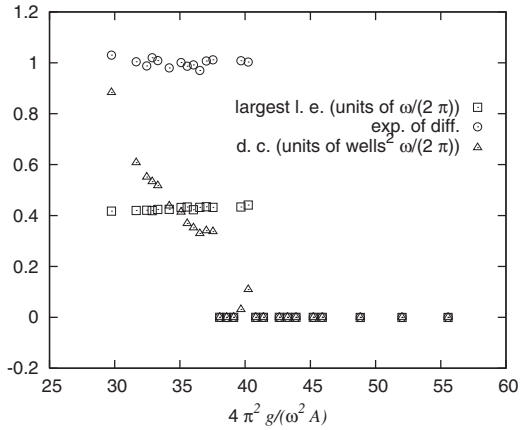

FIG. 4. Largest Lyapunov exponent (l.e.) (squares), and, for the diffusion in the wells, the diffusion exponent (circles) and the diffusion constant (d.c.) (triangles) are plotted as functions of $4\pi^2 g/(\omega^2 A)$. The parameters are $E=0.01$, $\alpha=0.999$, and $\beta=0.96$. The wells are $400A$ wide. The particle hops across a kink in the sawtooth for less than 1% of the hops. When the Lyapunov exponent is not positive, both the diffusion exponent and diffusion constant are zero.

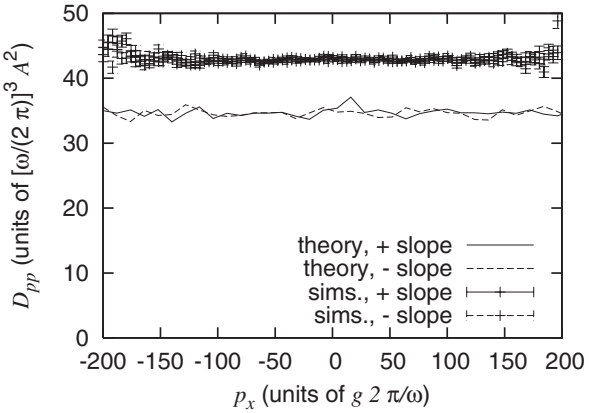

FIG. 5. D_{pp} as a function of the horizontal momentum for $E=0.05$, $4\pi^2 g/(\omega^2 A)=0.2083$, $\alpha=1$, and $\beta=0.5$. To leading order in the time-scale separation parameter, the diffusion does not depend on the momentum.

(18)–(23). However, the theoretical expression for the diffusion can be evaluated more accurately numerically.

In order to obtain numerical estimates for the diffusion and drift from the time-scale separation theory, it is necessary to define a trajectory for fixed slow coordinates, such as in Eq. (8). In a system with continuous dynamics there is no problem defining such a trajectory. However, in systems with discrete time, like the one under consideration here, after every finite time step the slow coordinates are mapped as well.

One cannot simply ignore this, as the change in the slow coordinates leads to a change in the excluded parts of phase space of the fast coordinates. Afterward y and p_y may not be consistent with the old x and p_x. Some combinations of horizontal and vertical coordinates are not allowed, because they might place the particle under the surface. The drastic nature of this problem is related to the hard interaction between the particle and the surface. Nevertheless, if the collisions were governed by a soft potential rather than a hard-core interaction, problems would still arise with conservation of the relative potential energy and a similar transformation would be necessary.

A. Transformation of variables

To deal with this problem a transformation of coordinates is necessary. After a suitable coordinate transformation the expression for the trajectory in Eq. (8) is well defined, and may be substituted into Eqs. (9) and (10).

The time-scale separation theory is not restricted to Hamiltonian systems, and in fact the bouncing-ball billiard is not Hamiltonian, and so a canonical transformation is not required. As the transformed fast coordinates one may use the position and momentum orthogonal to the surface in the reference frame of the moving surface.

$$\tilde{x} = x, \tag{26}$$

$$\tilde{p}_x = p_x, \tag{27}$$

$$\tilde{y} = \frac{y - h(x,t) \pm xE}{1 + E^2}, \tag{28}$$

$$\tilde{p}_y = \frac{p_y - v(x,t) \pm Ep_x}{1 + E^2}. \tag{29}$$

In the transformation of the fast variables, the terms involving the slow coordinates are of the order of the slope, which is of order ϵ. The transformation affects the functions $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ in Eqs. (11) and (12) by at most terms of order unity, and so the time-scale separation is not affected.

B. Diffusion

The integral expression for the diffusion in Eq. (9) can be evaluated numerically by calculating a trajectory of $\tilde{y}$ and $\tilde{p}_y$ for fixed horizontal position and momentum. Results for the diffusion in the horizontal momentum are plotted in Figs. 5 and 6 along with the numerical estimates for the same calculated for the full system. In the full system only parts of the trajectories were included where the particle does not hop across the center of the well, where the slope changes. The leading order of the diffusion in the horizontal momentum does not depend on the momentum or the sign of the slope of the profile. The slope in Fig. 6 corresponds to a quadratic dependence in E. The numerically evaluated theoretical prediction is quantitatively similar to the actual diffusion, and the predicted dependence on E, Eq. (20), is recovered. This is encouraging, because it suggests that the presence of periodic islands does not affect the diffusion and drift. The discrepancy between the theoretical prediction and the numerical results is probably due to the fact that the discrete dynamics of the bouncing-ball billiard lead to additional terms. However, the numerical similarity suggests that expressions along the lines of Eqs. (9) and (10) also exist for systems with discrete time.

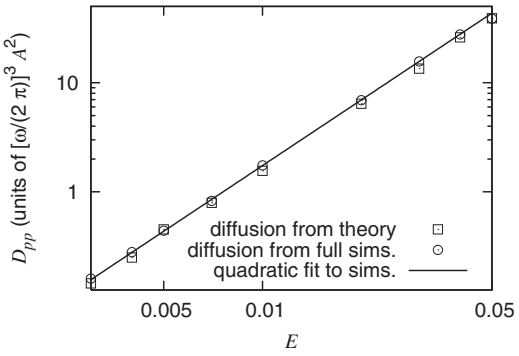

FIG. 6. D_{pp} as a function of E for $p_x=0.0$, $4\pi^2 g/(\omega^2 A)=0.2083$, $\alpha=1$, and $\beta=0.5$. The fit is quadratic and shows that the numerical results are consistent with the theory, Eq. (23).

C. Drift

The integral in the expression for the drift in Eq. (10) predicts a damping term that is of higher order in E. In order to confirm the presence of this term, one may consider a system without restitution parallel to the surface ($\alpha=1$). In such a system there is no first-order damping term in the drift, and so the second-order term appearing in Eq. (10) becomes the leading order. In Fig. 7, the average drift in the horizontal momentum is plotted as a function of the momentum for a specific set of parameters. The averaging was performed over all positions, with both negative and positive slope, but excluding any hops crossing the kinks in the sawtooth. The drift due to such hops is affected by the change in the slope and therefore contains a position-dependent term. The friction coefficient can be determined from the slope of the fit and is estimated at $(1.598\pm0.008)\times10^{-5}$. In Fig. 8, the friction coefficient is plotted as a function of ϵ for a set of systems with otherwise identical parameters. The dependence of the friction on the slope, proportional to E^2 from Eq. (23), is confirmed by the simulation results.

D. The absence of anomalous diffusion in the simulations

Near the transition between periodic orbits and chaos, a marginally unstable periodic orbit may exist. However, in order to sample the marginal instability, the linear (in)stability must vanish. This requires extremely accurate knowledge of the parameter values at the transition point. In a simulation such accuracy does not appear to be attainable. The system spends most of its time near a periodic orbit and exhibits intermittency. Nevertheless, the diffusion is normal and the orbits decay exponentially. Figure 9 shows an example of such an intermittent trajectory. The trajectories near the periodic orbit decay exponentially, but very slowly. Once the orbit has decayed sufficiently, the trajectory becomes chaotic and the particle starts to hop to other wells, sometimes even skipping wells completely. Eventually it comes back close to the same periodic orbit, as can be seen from the hopping times. The diffusion is normal, with an exponent of 0.993 ± 0.007.

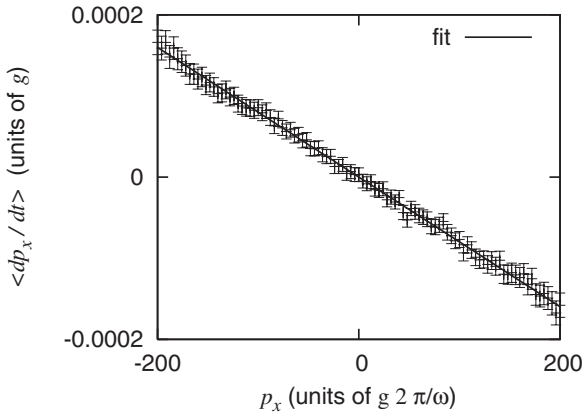

FIG. 7. Average drift in the horizontal momentum as a function of the horizontal momentum in a system without horizontal restitution. Parameters: $E=0.05$, $4\pi^2 g/(\omega^2 A)=0.2083$, $\alpha=1$, and $\beta=0.9$.

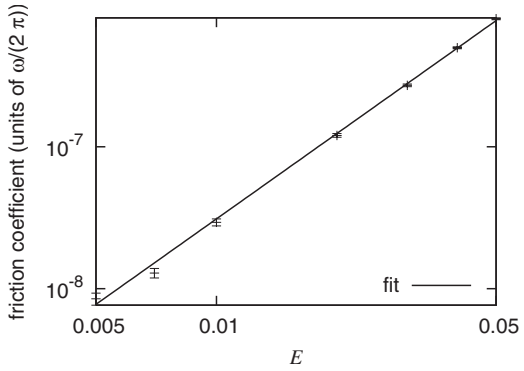

FIG. 8. Linear damping of the horizontal momentum in systems without horizontal restitution as a function of the coupling E. Note that the time-scale separation parameter ϵ equals the slope E. Apart from E, all parameters are as in Fig. 7. The fit corresponds to a second-order dependence on E.

VI. CONCLUDING REMARKS

For small slopes of the sawtooth height profile, the bouncing-ball billiard was shown to exhibit a time-scale separation between the horizontal and vertical coordinates. The bouncing-ball billiard is a system in discrete time. We showed that the slow dynamics can be interpreted as dynam-

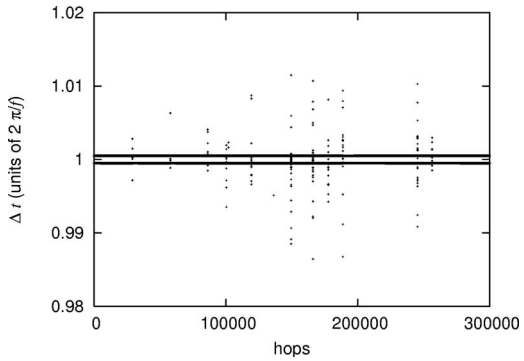

FIG. 9. Hopping time versus hop number is plotted for a trajectory of a system with parameters which produce an extremely weakly unstable periodic orbit of period 2. In this case $4\pi^2 g/(\omega^2 A)=47.655\,055$, $\alpha=0.99$, $\beta=0.5$, and $E=0.01$.

ics in continuous time, since the rate of change per discrete time step is very small. As a result, a Fokker-Planck-equation would be a suitable model for the reduced system.

Transport in the horizontal direction was shown to be related to the dynamics of the vertical coordinates. If the vertical coordinates are chaotic and correlation decays exponentially, the separation of time scales leads to normal diffusion in the horizontal coordinates. If the vertical coordinates are in a periodic regime, there is no diffusion. There is no anomalous diffusion for almost all combinations of parameters, and it is not straightforward to find anomalous diffusion in simulations. The results from the numerical simulations confirm the predictions from the time-scale separation theory and show that low-dimensional chaos is a source of stochastic motion.

The theoretical and numerical investigations of the sawtooth bouncing-ball billiard presented in this paper have raised questions with regard to the effects of the presence of stable islands in the fast subsystem.

We also demonstrated that the direct translation of the time-scale separation theory for time-continuous systems into the time-discrete case yields reasonable results. The numerical evaluation of the theoretical prediction for the diffusion as expressed in Eq. (9) and the diffusion calculated from simulations of the full system compare well with each other. This suggests that expressions similar to Eqs. (9) and (10) exist for systems with discrete time and that a corresponding theory for discrete time systems should be feasible.

Finally, if the profile is not symmetric, due to the fact that the bouncing-ball billiard is not a Hamiltonian system, and not in equilibrium, a total drift cannot be excluded. Therefore, it might be worthwhile to investigate ratchetlike profiles.

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